Technical Notes and Correspondence

Analysis of Pinning-Controlled Networks: A Renormalization Approach
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Abstract—In this paper, a renormalization approach is introduced for analyzing pinning-controlled networks. The renormalization process consists of two operations, edge weight and node reduction, and is built on a new concept of passivity comparison in the sense of Lyapunov V-stability. Furthermore, a cascaded model resulted from the renormalization process in a layer structure is presented for estimating the V-stability of a network. Finally, simulation studies are presented for illustration and verification of the theoretical results.

Index Terms—Complex network, layer, Lyapunov V-stability, passivity, pinning control, renormalization.

I. INTRODUCTION

Many large-scale systems in nature and human society can be described by networks with nodes representing individual entities and edges representing the connections or interactions among them. Typical examples include the Internet, World Wide Web, biological neural networks, scientific collaboration networks, transportation networks, and so on ([1], [2] and references cited therein). The ubiquitous appearance of various complex networks has attracted increasing attention of researchers from physics, biology, social science, computer science, mathematics and engineering communities.

One recent research focus is the analysis and control of dynamical behaviors of complex networks, including epidemic spreading [3], [4], traffic congestion and decongestion [5]–[7], and synchronization and desynchronization of coupled oscillators [8]–[13]. In particular, regulating dynamical behaviors of a complex network by means of control has gradually become a focal research topic among control scientists and engineers, for which the approach of pinning control is especially effective [14], [15]. Pinning control applies local feedback injections to a small fraction of nodes on a large-size network, thereby achieving some intended global performances over the entire network. Wang and Chen [14] showed that, due to the extremely inhomogeneous connectivity distribution of scale-free networks, it is much more effective to pin some most-highly connected nodes than to pin randomly selected nodes. Further investigation by Li et al. [15] showed that pinning control of a general network via virtual control has special effects on the spreading dynamics. Chen et al. [16] proved that, if the coupling strength is large enough, even one single pinning controller is able to control a large network. Sorrentino et al. [17] developed some special techniques for analyzing the pinning-controllability of various complex networks. More recently, Xiang and Chen [18] proposed a V-stability scheme based on the classical Lyapunov stability theory. Under the V-stability scheme, pinning control can be regarded as changing the passivity-degree values of the controlled nodes, and the stability of the entire controlled network can be determined by a characteristic matrix derived.

Since pinning control algorithms apply local feedback control actions only on a small fraction of nodes of the network, choosing suitable nodes to pin is clearly key to the success of this control strategy. To that end, one must have an effective method to analyze the stability or other important dynamical properties of the pinning-controlled network. However, most control methodologies reported thus far are based on global collective information about the whole network, which is infeasible for complex networks of large sizes in reality. In this technical note, under the V-stability scheme, a novel renormalization method consisting of two operations on nodes and edges is presented for effective analysis of pinning-controlled networks which can be very large in size. The renormalization process is built on the new concept of passivity comparison with a distinguished feature that only local information is used. Furthermore, with the new concept of layer structure, a cascaded model is derived from the pinning-controlled network by the renormalization process, so that if the pinning-controlled network is V-stable then so is its cascaded model, which will be very useful in pinning control design for general complex networks.

The rest of this technical note is organized as follows. In Section II, some preliminaries about the problem formulation are given. Then, in Section III, the new renormalization process is introduced based on the concept of passivity comparison. The cascaded model for estimating the V-stability of the network is proposed with a layer structure in Section IV. Finally, in Section V, simulation studies are presented for illustration and verification of the theoretical results.

II. PRELIMINARIES

A. Pinning Control

Consider a dynamical network consisting of $N$ linearly and diffusively coupled identical nodes, with each node representing an $n$-dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} a_{ij} \Gamma(x_j - x_i), \quad i = 1, \cdots, N \quad (1)$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ is the state vector of node $i$, $\Gamma = \text{diag}(\tau_{ir}) \in \mathbb{R}^{n \times n}$ is the inner-linking matrix, in which $\tau_{ii} \neq 0$ means that two nodes are linked through their $i$th components of the corresponding state vector, $l = 1, \cdots, n$. The same $\gamma$ for all nodes is restrictive in mathematics but is ubiquitous in many real networks, such as artificial internet, natural transduction networks in cell, and so on. Moreover, the weighted coupling matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ represents the configuration of the undirected network: if there is a connection between node $i$ and node $j$ ($i \neq j$), then $a_{ij} = a_{ji} \neq 0$; else, $a_{ij} = a_{ji} = 0$ ($i \neq j$); with the diagonal elements being $a_{ii} = -k_i$, where $k_i$ is the weighted degree of node $i$, $i = 1, \cdots, N$. Due to this diffusive connection, i.e., $\sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji} = k_i, i = 1, \cdots, N$, the coupling matrix $A$ defined above is a negative semi-definite and
symmetric matrix. Suppose that there are no isolated clusters in the network; thus, the matrix $A$ is also irreducible [21].

The objective of pinning control in this technical note is to stabilize network (1) by applying controllers on a small fraction $\delta (0 < \delta \ll 1)$ of nodes, to a stationary state $\bar{x}$ satisfying

$$x_i \rightarrow \bar{x} \quad \text{and} \quad f(\bar{x}) = 0, \quad i = 1, \cdots, N. \quad (2)$$

Without loss of generality, assume that the nodes $x_1, x_2, \cdots, x_l$ are selected for pinning, with $l \leq [\delta N]$, where $[\cdot]$ represents the integer part of the real number $\delta$. Thus, the controlled network can be described by

$$\dot{x}_r = \begin{cases} f(x_r) + \sum_{j=1}^{N} a_{rj} \Gamma(x_j - x_r) + u_r, & r = 1, \cdots, l, \\ f(x_r) + \sum_{j=1}^{N} a_{rj} \Gamma(x_j - x_r), & r = l + 1, \cdots, N \end{cases} \quad (3)$$

with controllers $u_r$ to be designed, $r = 1, 2, \cdots, l$.

**B. Lyapunov V-Stability**

In [18], the concept of Lyapunov V-stability was introduced and used to convert the stability problem of a dynamical network to a simple test on the negative definiteness of a characteristic matrix of the network. This scheme is briefly reviewed here for completeness of the presentation.

**Assumption 1:** For network (1), there is a Lyapunov function $V(x_i) : \mathcal{D} \rightarrow \mathbb{R}_+$ with $V(\bar{x}_i) = 0$ such that the inequality

$$\frac{\partial V(x_i)}{\partial x_i} (f(x_i) - \theta_i \Gamma(\bar{x}_i - x_i)) < 0, \quad \forall x_i \in \mathcal{D}, x_i \neq \bar{x}_i \quad (4)$$

holds for a scalar $\theta_i$, where the domain $\mathcal{D} \subseteq \mathbb{R}^n$ is a compact set containing the equilibrium point $\bar{x}_i$ of the network.

Here, $\theta_i$ is called the passivity degree (PD), representing the passivity level of the node dynamics $f(x_i)$ with respect to $\bar{x}_i$. Intuitively, $\theta_i < 0$ means that the node needs additional energy from outside to become stable, while $\theta_i > 0$ means that the node itself is stable and meanwhile can provide some energy to stabilize other nodes in the network.

A quadratic Lyapunov function is used here, in the form of $V(x) = x^T \bar{Q} x$ with a positive definite matrix $\bar{Q}$ satisfying

$$\bar{Q} \Gamma + \Gamma \bar{Q} \geq 0. \quad (5)$$

Throughout this technical note, the notation $A(\leq) < 0$ means that the matrix $A$ is negative (semi-) definite. Once the PD value $\theta$ is confirmed, the V-stability of a network can be judged by its characteristic matrix (detailed by Th. 2 and Th. 4 in [18]). The characteristic matrix of network (1) is

$$C = -\theta I_N + A \quad (6)$$

where $I_N$ denotes the unit matrix of order $N$. Under the V-stability scheme, the control input $u_r$ is regarded as a force to change the PD value of the controlled node, from $\theta$ to $\theta + d_r$, for some scalar $d_r$, namely

$$\frac{\partial V(x)}{\partial x}(f(x) + u_r(t) - (\theta + d_r) \Gamma(\bar{x} - x)) < 0, \quad \forall x \in \mathcal{D}, x \neq \bar{x} \quad (7)$$

where $r = 1, \cdots, l$. Thus, the characteristic matrix of the closed-loop network (3) is

$$C_{c,\theta+d_r} = -\theta I_N + A - D \quad (8)$$

where $D$ is a diagonal matrix with $l$ controlled elements being $d_r$ while the remaining elements being zero. If the characteristic matrix $C_{c,\theta+d_r}$ is negative semi-definite, then the pinning controlled network is locally stable or globally stable depending on the range of the concerned domain $\mathcal{D}$ of (4).

V-stability scheme divides the pinning control problem into two parts: one is the control law design, namely the format of $u_r$, which changes the PD value of the controlled subsystem (7); the other is the pinning strategy design, namely the selection of the controlled nodes, which makes the characteristic matrix be negative semi-definite. The former is the typical problem in classic control theory and can be solved by existing methods, therefore is not further discussed here; while the latter is related to network properties and requires new insights, therefore is further discussed below in this technical note.

**III. Renormalization Process**

**A. Passivity Comparison**

In terms of the V-stability property, every network has its own characteristic matrix, (6) or (8), based on which one can define some comparable measures, between two networks, as follows:

Given two networks $A$ and $B$ with characteristic matrices $C_A$ and $C_B$, respectively, if $C_A \leq 0 \Rightarrow C_B \leq 0$, then network $A$ is said to be passively weaker than network $B$; if $C_B \leq 0 \Rightarrow C_A \leq 0$, then network $A$ is said passively stronger than network $B$; if both of them hold simultaneously, then network $A$ is said to be passively equivalent to network $B$.

The comparison between any pair of networks is based on their characteristic matrices. For two networks with given characteristic matrices, they can be compared so as to arrive at a unique conclusion, either passively stronger, passively weaker, passively equivalent, or undetermined.

The significance of constructing this comparison lies in that in the study of the stability of a complex network, one can deal with another network that is simpler but passively weaker than the original one, thereby a final stability conclusion about the original network can be reached, although rather conservatively but certainly safely. From a physical viewpoint, network $A$ is passively weaker than network $B$ means that if a controller can be designed to stabilize network $A$ then this controller can also stabilize network $B$, but of course the converse is not true in general.

**B. Renormalization Process**

Two operations, edge weighting and node reduction, are now introduced to change the topology of a network under certain logical relation with the V-stability. This is referred to as the renormalization process in this technical note.

**Theorem 1:** Given a network $A$, change the edge weight by $\gamma \in [-\sigma_A, \infty]$ between node $i$ and node $j$ to yield a new network $B_{ij}$, keep the ordering of all nodes and also keep the PD values and connected edges of all nodes unchanged, except that the PD values of node $i$ and node $j$ are increased by $\delta_i \in \mathbb{R}$ and $\delta_j \in \mathbb{R}$, respectively. If

$$\begin{cases} \gamma \geq \max\{\delta_i, \delta_j\} \\ \gamma^2 \leq (\gamma - \delta_i)(\gamma - \delta_j) \end{cases} \quad (9)$$

then network $B_{ij}$ is passively stronger than network $A$; while if

$$\begin{cases} \gamma \leq \min\{\delta_i, \delta_j\} \\ \gamma^2 \leq (\gamma - \delta_i)(\gamma - \delta_j) \end{cases} \quad (10)$$

then network $B_{ij}$ is passively weaker than network $A$. 

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Proof: The two characteristic matrices of network $A$ and network $B_{ij}$ are described respectively by,

$$C_A = \begin{bmatrix}
    a_{i1} - \theta_1 & \cdots & a_{ij} \\
    \vdots & \ddots & \vdots \\
    a_{ij} & \cdots & a_{j1} - \theta_j
\end{bmatrix}$$

$$C_{B_{ij}} = \begin{bmatrix}
    a_{i1} - \gamma - \theta_1 + \delta_i & \cdots & a_{ij} + \gamma \\
    \vdots & \ddots & \vdots \\
    a_{ij} + \gamma & \cdots & a_{j1} - \gamma - \theta_j + \delta_j
\end{bmatrix}$$

(11)

where the blank part and the "..." part are the same for both matrices, which are not important in this discussion. To prove that network $B_{ij}$ is passively stronger (weaker) than network $A$, it suffices to show that $C_{B_{ij}} - C_A \preceq (\geq 0)$. Note that this inequality is equivalent to

$$\begin{bmatrix}
    -\gamma + \delta_i & \gamma \\
    \gamma & -\gamma + \delta_j
\end{bmatrix} \preceq (\geq 0)$$

(12)

which is satisfied under the condition (9), (10), based on the Schur complement lemma.

The following result is a consequence of this theorem.

Corollary 2: The network with edge-weight reduction is passivity weaker than the original one; while the network with edge-weight increase is passivity stronger than the original one.

Proof: They are the special cases of edge weighting with $\delta_i = 0$, and $\gamma < 0$ and $\gamma > 0$, respectively.

Theorem 3: For a node $p$ in a given network $A$ with $m$ connections to the neighboring nodes $q_1, \ldots, q_m$, let its PD value be $\theta_p$. The PD values and coupling strengths of the neighboring nodes $q_j$ of $p$ are denoted by $\theta_{q_j}$ and $a_{pq_j}$ respectively, $i = 1, \ldots, m$. Removing node $p$ and its connected edges yields a new network $B_p$. Keep the original ordering and PD values of all nodes unchanged, except that the PD values of nodes $q_i$ are increased by $\delta_i$, and the coupling strengths between node $q_i$ and node $q_j$ are increased by $\gamma_{ij}, i, j = 1, \ldots, m$. If

$$\sum_{i=1}^{m} a_{pq_i} + \theta_p > 0$$

(13)

$$\delta_i = \frac{\sum_{i=1}^{m} a_{pq_i}\theta_p}{a_{pq_i}}, \quad i = 1, \ldots, m$$

(14)

$$\gamma_{ij} = \frac{\sum_{i=1}^{m} a_{pq_i}a_{q_jq_k}}{a_{pq_i}}, \quad i, j = 1, \ldots, m$$

(15)

then network $B_p$ is passively equivalent to network $A$.

Proof: Without loss of generality, let node $p$ be the last one in network $A$. Thus, the characteristic matrices of the two networks are

$$C_A = \begin{bmatrix} a_{11} & & \\ \vdots & \ddots & \vdots \\ a_{mm} & & a_{1m} \end{bmatrix}$$

$$C_{B_p} = \begin{bmatrix} a_{11} + \delta_1 & \cdots & a_{1m} + \delta_m \\ \vdots & \ddots & \vdots \\ a_{mm} & \cdots & a_{mm} + \delta_m \end{bmatrix}$$

$$W_{11} = \begin{bmatrix} 0 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & 0 \end{bmatrix}$$

$$W_{12} = \begin{bmatrix} a_{pq_1} & 0 & \cdots & 0 \\ 0 & a_{pq_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{pq_m} \end{bmatrix}$$

$$W_{2} = \begin{bmatrix} b_1 & \gamma_{i_1} & \cdots & \gamma_{i_m} \\ \gamma_{i_1} & b_1 & \cdots & \gamma_{i_m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{i_1} & \gamma_{i_1} & \cdots & b_m \end{bmatrix}$$

$$U \in \mathbb{R}^{(N_A-1) \times (N_A-1)}$$

denotes the invariant part with $N_A$ being the size of network $A$, $a_{ij} = -a_{pq_i} - \theta_{q_j}$, and $b_i = -(\theta_{q_j} + \delta_i) - \sum_{j \neq i} a_{pq_j} \gamma_{ij}$.

$$C = \begin{bmatrix} 0 & a_{pq_1} & \cdots & a_{pq_m} \\ a_{pq_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{pq_m} & 0 & \cdots & 0 \end{bmatrix}$$

(17)

which is nothing but $C_{B_p} \leq 0$ since $-a_{pq_i} + \sum_{i=1}^{m} \gamma_{ij} = -\delta_i$.

Theorem 3 reveals an interesting renormalization mechanism: One can remove any node (subsequently its connected edges), if the node being removed satisfies $\sum a_{pq_i} + \theta_p > 0$, from a given network thereby redistributing its PD value to its neighbors, by a quantity of $a_{pq_i}(\theta_p/(\sum a_{pq_i} + \theta_p))$, and strengthening the coupling strength between any pair of its connected neighboring nodes, by a quantity of $a_{pq_i}a_{q_jq_k}/(\sum a_{pq_i} + \theta_p)$. The new network so obtained is passively equivalent to the original one (Fig. 1). Note that the condition $\sum a_{pq_i} + \theta_p > 0$ is a natural constraint, since if this condition is violated then it is impossible to make a characteristic matrix be negative semi-definite.

Remark 1: When the edge weighting process is executed, the nodes connected by the weighted edge updates the passivity degree of itself $\delta_i$, and its connection weights $\gamma$. When node reduction process is executed, the reduced node sends $\delta_i$, and $\gamma_{ij}$, which can be calculated on the local information of the reduced node, to its neighboring node $q_i$, and then each $q_i$ updates the passivity degree of itself, and its neighboring nodes and connection weights. Therefore, it can be concluded that only local information is needed for the renormalization process of the whole network. It is also clear that parallel computation can be used to speed up the renormalization process if preferred.

Corollary 4: A controlled network is $V$-stable, i.e., its characteristic matrix is negative semi-definite, if and only if all the uncontrolled nodes can be successively renormalized by the operation of node reduction such that finally only the controlled nodes remain.

Proof: First, it can be shown that all diagonal elements of the characteristic matrix of a $V$-stable network are negative, except for those corresponding to the controlled nodes. If not, without loss of generality assume that $\sum a_{ij} + \theta_i = 0$ and $a_{12} > 0$ since no isolated node exists in the network. Thus, the characteristic matrix can be written as

$$C = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1m} \\ a_{pq_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{pq_m} & 0 & \cdots & 0 \end{bmatrix}$$

Fig. 1. (Color online) Renormalization of node reduction: node $p$ in the network on the left is removed, resulting in the network on the right, which is passivity equivalent to the original network on the left.
It is easy to verify that there is a real vector \( v = [v_1, v_2, 0]^T \in \mathbb{R}^N \) such that \( v^T C v > 0 \) where scalars \( v_1 \) and \( v_2 \) satisfy \( 2v_1v_2a_{12} - v_1(\sum a_{2j} + \theta_2) > 0 \). This is a contradiction to the semi-negative definiteness of the characteristic matrix. Therefore, all the diagonal elements corresponding to the uncontrolled nodes must be negative. Consequently, the node reduction operation can be carried out. Theorem 3 shows that the renormalization process of node reduction preserves the \( V \)-stability, as well as the property that there are no isolated nodes. Thus, one can perform this process successively, until all the uncontrolled nodes are being renormalized and only those controlled nodes are left out.

Conversely, since the control law can be designed freely, so that the PD values of the controlled nodes can be changed arbitrarily, the renormalized network that contains only the controlled nodes is obviously \( V \)-stable under suitably designed controllers. By Theorem 2, the renormalization process of node reduction does not change the \( V \)-stability of the network, therefore the original network is also \( V \)-stable.

IV. CASCADeD MODEL

A. Layer Structure

The set of nodes having the same distance\(^1\) to the controlled nodes compose a layer.

With the first layer being the initial set of controlled nodes, a controlled network can be viewed as a multi-layered structure on which a spreading process of (virtual) control power will take place. The control power from the first layer of nodes has been stabilized, first influences the nodes in the second layer which directly link with the controlled nodes. Then, through the nodes in the second layer the extra control power, after the second layer of nodes have been stabilized, is transferred to the nodes in the third layer, and so on. Fig. 2 visualizes this control power spreading process, which shows that each node is related only to the nodes located in the previous layer, the same layer, or the next layer, but not those located in any other layers.

Assume that there are \( r \) layers in a controlled network, formed in the above-described manner. By reordering the nodes in the layer they belong to, the coupling matrix \( A \) can be written as

\[
A = \begin{bmatrix}
A_{11} & A_{12} & 0 & \cdots & 0 \\
A_{12} & A_{22} & 0 & \cdots & 0 \\
0 & A_{21} & A_{22} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{r-1r}
\end{bmatrix}
\]

where \( A_{i(i+1)} \) denotes the connections between nodes in the \( i \)th layer and nodes in the \((i+1)\)st layer, therefore \( A_{ij} = A_{ji} \) denotes the coupling relations between nodes within the same \( i \)th layer, \( i = 1, 2, \ldots, r \).

Denote by \( \theta_{ij} \) the PD value of the \( j \)th node in the \( i \)th layer, and let

\[
\Theta_i = \text{diag}(\theta_{i1}, \ldots, \theta_{ii})
\]

with \( I_i \) being the number of nodes in the \( i \)th layer which satisfy \( \sum_{j=1}^{I_i} l_j = N \). Since the controlled nodes are all located in the first layer, one has \( D = \text{diag}(D_1, 0, \ldots, 0) \) with \( D_1 = \text{diag}(d_1, \ldots, d_1) \). Thus, the closed characteristic matrix of the controlled network (3) can be written as

\[
C_{\epsilon_{i,\epsilon_{r+1}}} = \begin{bmatrix}
A_1 - \Theta_1 - D_1 & A_{12} & \cdots & 0 \\
A_{12} & A_2 - \Theta_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{r-1r}
\end{bmatrix}
\]

(19)

Let \( \mathcal{N}_p \) and \( \mathcal{L}_q \) denote the index set of neighboring nodes of node \( p \) and the index set of the nodes in layer \( q \), respectively. Associated with the layer structure, the following notations are introduced.

**Node input weight.** \( \xi_{ij} = \sum_{k \in \mathcal{N}_p} a_{ij} \xi_k \): the sum of coupled weights of the \( j \)th node in the \( i \)th layer from the nodes in the \((i-1)\)st layer.

**Layer input weight.** \( \Xi_i = \sum_{j \in \mathcal{L}_q} \xi_{ij} \): the sum of node input weight of all the nodes in the \( i \)th layer.

**Layer passivity degree.** \( \Lambda_i = \sum_{j \in \mathcal{L}_q} \xi_{ij} \): the sum of the PD of nodes in the \( i \)th layer.

**Remark 2:** Although the considered network is bidirectional, with the layer structure it can be seen as a directed network to depict the spreading process of control power, as shown by Fig. 2.

**Theorem 5:** If a controlled network is \( V \)-stable, then the negative value of layer input weight \( -\Xi_i \) for the \( i \)th layer, excluding the first layer, is less than the sum of the PD values of the whole network excluding those nodes in the previous \((i-1)\) layers, \( i = 2, \ldots, r \).

**Proof:** If \( C \leq 0 \), then it follows from (19) that for the \( i \)th layer

\[
\begin{bmatrix}
A_i - \Theta_i & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_r - \Theta_r
\end{bmatrix}
\leq 0,
\]

\( i = 2, \ldots, r \) (20)

Separating the parts of the input links of the nodes on the \( i \)th layer, the above inequality can be rewritten as

\[
\begin{bmatrix}
W_i \quad -\Theta_i \\
\vdots & \ddots & \vdots \\
0 & \cdots & -\Theta_r
\end{bmatrix} +
\begin{bmatrix}
\Theta_i \\
\vdots \\
\Theta_r
\end{bmatrix}
\leq 0
\]

(21)

where the matrix \( W_i = \text{diag}(-\xi_1, \ldots, -\xi_i) \), \( i = 2, \ldots, r \).

Noting that \( a_{ii} = -\sum_{j \neq i} a_{ij} \), the vector \((1, 1, \ldots, 1)^T\) belongs to the null space of matrix \( A_{r+1}^T \). Also noting that \( \xi_{ii} \) is the row sum of matrix \( A_{r+1}^T \), it follows that:

\[
\begin{bmatrix}
A_{(i+1)r}^T & A_i & A_{(i+1)} & \cdots & 1, 1, \ldots, 1
\end{bmatrix}^T = W_i
\]

Therefore, it can be obtained that the last term of the left hand of the above inequality has a zero eigenvalue associated with the eigenvector
and coupling

strength

two renormalization steps:

one, the critical (bottleneck) point is clearly

index numbers of layers increase. Since the first layer is the controlled

network can be represented by a cascaded model (see Fig. 3),

nodes should be ignored when the control power on each controlled

weight of the second layer. The larger the

study (not shown here due to the space limitation), to maximize

with highest degrees to be the controlled nodes [15]. In our simulation

Theorem 5 that it is better to order the nodes weights decreasingly as the

index numbers of layers increase. Since the first layer is the controlled

one, the critical (bottleneck) point is clearly \( \Xi_2 \), which is the layer input

weight of the second layer. The larger the \( \Xi_2 \), the bigger the range of the

permissible coupling strength for achieving network stabilization.

This is slightly different from the earlier conclusion, selecting nodes

with highest degrees to be the controlled nodes [15]. In our simulation

study (not shown here due to the space limitation), to maximize \( \Xi_2 \) is

somewhat better than maximizing the degrees of the controlled nodes.

Intuitively, the node degree due to the connections between controlled

nodes should be ignored when the control power on each controlled

cascaded model is sufficiently large.

B. Cascaded Model

At the layer level where each layer is regarded as a node, any

controlled network can be represented by a cascaded model (see Fig. 3),

which is constructed as follows:

- Keep the first layer (controlled nodes) unchanged and replace the

  remaining layers by nodes with each PD value being the layer

  passivity degree \( \Lambda_i \), \( i = 2, \ldots, r \).

- Let the edges of controlled nodes be unchanged (with another end

  connecting the node, which represents the second layer), and let

  the remaining edges be bidirectional and be weighted by the corre-

  sponding layer input weights.

Theorem 6: A controlled network is passively weaker than its cas-

caded model.

Proof: To prove this, we show how to perform the renormaliza-

tion process on the controlled network to ultimately obtain its cascaded

model.

Take a node in the layer as the terminal node, denoted as \( q_i \). For any

remaining node in the same layer, denoted as \( p \), perform the following

two renormalization steps:

R1. Enlarge the weight between node \( q_i \) and node \( p \), to infinity,

with both their PD values unchanged;

R2. Remove the node \( p \) by the node reduction process.

Fig. 3. (Color online) Renormalization process converts the original network to a cascaded model, where only one node (in the first layer) is controlled. Each layer is finally reduced to one single (big) node with PD value \( \Lambda_i \) and coupling strength \( v_{ij}, i = 1, \ldots, r \).

Denote the original network, the middle network and the last network in

Fig. 4 by \( N_{100} \), \( N_{R1} \) and \( N_{R2} \), respectively. According to Corollary

2, the increasing weight step R1 makes the original network \( N_{100} \) be

passively weaker than network \( N_{R1} \). In the case that \( a_{1p} \) is infinity, the

(14) and (15) reduce to

\[
\delta_{1t} = \theta_{p}, \quad \delta_{i} = 0, \quad \forall i \in \mathcal{N}_t / \{q_t\} \tag{22}
\]

and

\[
\gamma_{aq_{tj}} = a_{pi}, \quad \gamma_{q_{tj}} = 0, \quad \forall i, j \in \mathcal{N}_t / \{q_t\} \tag{23}
\]

respectively, where \( \mathcal{N}_t \) denotes the index set of all neighboring nodes of

node \( p \). Thus, after the above two steps, the PD value of the reduced

node is moved to the terminal node and the original edges connected

with node \( p \) change their directions to the terminal node \( q_t \), detailed in

Fig. 4. Since the R2 step preserves the V-stability, network \( N_{R1} \) is

passively weaker than network \( N_{R2} \). By executing the two steps R1 and

R2 on all the nodes, except for the terminal node in each layer, the

cascaded model will finally be derived.

Remark 3: Different from the renormalization process, which can be

operated on the local information. The layer structure and the cascaded

model derived on layers needs the central information. Of interest is,

as shown by Theorem 6, that the centralizing cascaded model can be
derived from some distributed renormalization processes.

Remark 4: To study a controlled network, one can conveniently

investigate its corresponding cascaded model instead. If its cascaded

model cannot achieve the V-stability, then neither can the original net-

work. Of course, the converse may not be true in general.

V. SIMULATION STUDY

For demonstration, the chaotic Lorenz system is used as the nodes

in a network. A single Lorenz system is described by [20]

\[
\begin{align*}
\dot{x}_1 &= \alpha (x_2 - x_1) \\
\dot{x}_2 &= \gamma x_1 - x_2 - x_1 x_3 \\
\dot{x}_3 &= \beta x_1 x_3 + x_1 x_2.
\end{align*}
\tag{24}
\]

When \( \alpha = 10, \beta = 8/3 \) and \( \gamma = 28 \), this Lorenz system has a chaotic

attractor, with complicated dynamics. Assume that the inner-linking

matrix \( \Gamma = \text{diag}(0, 1, 0) \); that is, the connection is through the second

variable \( x_2 \). The entire dynamical network is described by

\[
\begin{align*}
\dot{x}_{1i} &= \alpha (x_{i2} - x_{1i}) \\
\dot{x}_{2i} &= \gamma x_{1i} - x_{2i} - x_{1i} x_{3i} + c \sum_{j=1}^{N} a_{ij} x_{j2} \\
\dot{x}_{3i} &= -\beta x_{1i} x_{3i} + x_{1i} x_{2i}, \quad i = 1, \ldots, N.
\end{align*}
\tag{25}
\]

Here, the coupling matrix is expressed by \( cA \) with \( c \) representing the

uniform coupling strength and \( A = [a_{ij}] \) being a 0-1 matrix describing
the network topology. In this setting, the uniform coupling strength $c$ is a convenient index for the comparison in the following simulations.

The objective is to control by pinning the network to the nontrivial equilibrium $\bar{x} = (\sqrt{\frac{72}{7}}, \sqrt{\frac{72}{27}}, 1)^T$ of the Lorenz system.

Choose the Lyapunov function $V(x_i) = \sum_{j=1}^{n} (x_{ij} - \bar{x}_j)^2$. It can be verified that the permissible PD value of the Lorenz system in the so-coupled network satisfies $-\theta > (121/13) - 1$ over the whole domain of $D = \mathbb{R}^n$. Take $\theta = -8.4$. This implies that in such a connection, the Lorenz-system nodes need external energy to converge to the equilibrium $\bar{x}$. The simple state-feedback controller $u = d(\bar{x}_2 - x_2)$ is applied to the second variables of a small fraction of nodes. It is easy to see that the PD value of each controlled node will be changed by $d$, the controller gain value.

For a network, with given node PD values, control gains, and controlled nodes, there is a minimum value of the coupling strength $c^*$ such that the network is $V$-stable for all $c > c^*$ or else it is not $V$-stable. It can be seen from (8) (where $A$ is now $cA$) that $c^*$ can be calculated by the LMI toolbox. In our simulation study, the corresponding resultant cascaded models for both a BA scale-free network and a random network were tested by calculating their $c^*$ values.

Consider a network of 100 Lorenz-system nodes. Randomly generate an integer $0 < l_1 \leq 100$, and then choose $l_1$ controlled nodes to calculate $c^*$. Denote the minimum coupling strengths of the original network and of the cascaded model by $c_1^*$ and $c_2^*$, respectively. Totally 300 trials have been run for each network structure. The results are shown in Fig. 5 and Fig. 6, respectively, where for clarity they are sorted in the non-decreasing order of $c_i^*$.

It is clear that $c_1^* \geq c_2^*$ always holds for both cases. This is consistent with the analytic results given above. It can be seen that the $c^*$ error between the original network and its cascaded model for the random network is much smaller than that for the BA scale-free network. This can be explained as follows: Intuitively, it is the node with minimum weighted degree that determines the minimum feasible coupling strength $c^*$. The $R1$ step to make two nodes connected by an infinite weight edge causes these two nodes to have a maximum weighted degree. The random network is a homogenous network, so roughly all the nodes have about the same degree and symmetry. Therefore, the $R1$ step in constructing its cascaded model, which leads to $c_1^* \geq c_2^*$, can be neglected in a random network, since the change on the minimum weighted degree is very small after the $R1$ step. But the BA scale-free network is inhomogeneous networks, where the change on the minimum weighted degree caused by $R1$ step is so strong that the difference between $c_1^*$ and $c_2^*$ is prominent and has significant impact on the network stability.

**VI. CONCLUSION**

In this technical note, two new concepts for analyzing pinning-controlled networks have been introduced, i.e., passivity comparison and layer structure, under the $V$-stability framework. Based on passivity...
comparison, a renormalization process consisting of two kinds of elementary operations, i.e., edge weighting and node reduction, has been presented and analyzed. The renormalized network has a logical relation with the original one in the sense that the former is easier to be stabilized than the latter. Layer structure, on the other hand, shows that the control power is virtually spread out layer by layer from the controlled nodes to the entire network. A cascaded model, which is a simplified model of the network at the layer level, has been presented to test the $V$-stability of the controlled network. Finally, some simulation examples have been provided to illustrate and verify the correctness and effectiveness of the theoretical results.

REFERENCES


Stability Optimization of Hybrid Periodic Systems via a Smooth Criterion

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Abstract—We consider periodic orbits of controlled hybrid dynamic systems and want to find open-loop controls that yield maximally stable limit cycles. Instead of optimizing the spectral or pseudo-spectral radius of the monodromy matrix $A$, which are non-smooth criteria, we propose a new approach based on the smoothed spectral radius $\rho_\beta(A)$, a differentiable criterion favorable for numerical optimization. Like the pseudo-spectral radius, the smoothed spectral radius $\rho_\beta(A)$ converges from above to the exact spectral radius $\rho(A)$ for $\beta \rightarrow 0$. Its derivatives can be computed efficiently via relaxed Lyapunov equations. We show that our new smooth stability optimization program based on $\rho_\beta(A)$ has a favorable structure: it leads to a differentiable nonlinear optimal control problem with periodicity and matrix constraints, for which tailored boundary value problem methods are available. We demonstrate the numerical viability of our method using the example of a walking robot model with nonlinear dynamics and ground impacts as a complex open-loop stability optimization example.

Index Terms—Eigenvalue optimization, Lyapunov equation, periodic orbits, robotic motion, robustness, smoothed spectral radius, stability.

I. INTRODUCTION

Stability optimization of nonlinear periodic systems with hybrid dynamics is a difficult but very important task. It arises when a technical system is best operated periodically and has to be controlled in such a way that its cyclic steady state or periodic orbit is stable, robust against perturbations, and optimized with respect to certain desirable features. A typical example is human or robotic running, where the periodic motion has to be robustly stable and allow the runner to move as fast as possible. The dynamics of running or hopping are often described by hybrid dynamics due to the ground impacts.

Other examples are periodically operated simulated moving bed (SMB) processes [26], looping kites [15], or iterative feedback tuning with time and frequency domain constraints [3], [7].

Periodic systems can be stabilized in two ways: one based on sensors, actuators, and feedback control, the second based on intrinsically open-loop stable orbits [19], [20]. Here, we treat the second, where system parameters—for example limb lengths in the case of a running robot—and time varying inputs, or control functions—for example periodic torque commands—are simultaneously optimized to yield inherently stable periodic orbits. We note that the first case—feedback

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