Pinning Control of Uncertain Complex Networks to a Homogeneous Orbit

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Abstract—A sufficient condition for the stability of uncertain complex networks is derived in terms of linear matrix inequalities based on the V-stability tool, which associates the self-dynamics of nodes with passivity degrees. Then, a pinning control strategy is proposed on the developed condition to stabilize the uncertain complex networks to a homogenous orbit. As an illustrative example, a network with the Lorenz system as node self-dynamics is simulated to verify the analytic results.

Index Terms—Pinning control, uncertain complex networks, V-stability.

I. INTRODUCTION

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VER the past decades, growing interests have been shown in complex networks since more and more complex systems in nature or society have been modeled by them. A focal point of interests in recent studies has been to analyze and discuss the collective behaviors of complex networks, such as fireflies flashing, crickets chirping, heart cells beating [1], and formation of marine surface craft [2]. One basic stability problem in the field of control is determining if such collective behaviors are attractive or stable. There are a lot of control problems proposed for solving it in the literatures, among which behaviors are attractive or stable. There are a lot of control problem in the field of control is determining if such collective and formation of marine surface craft [2]. One basic stability as fireflies flashing, crickets chirping, heart cells beating [1], and discuss the collective behaviors of complex networks, such systems in nature or society have been modeled by them. A

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II. UNCERTAIN COMPLEX DYNAMICAL NETWORK MODEL

Consider a dynamic complex network consisting of \( N \) identical nodes with uncertain diffusive couplings, with each node being an \( n \)-dimensional dynamical system. The state equation of the network is described by

\[
\dot{x}_i = f(x_i) + h_i(x_1, x_2, \ldots, x_N) + \sum_{j=1 \atop j \neq i}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i), \quad i = 1, 2, \ldots, N
\]

where \( x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n \) is the state vector of node \( i \), \( f(x_i) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) represents the self-dynamics of node \( i \), and \( h_i(x_1, x_2, \ldots, x_N) \) denotes the unknown couplings of each node; on the other hand, the last part denotes the known linear couplings, where constant \( c_{ij} \) is the coupling strength between node \( i \) and node \( j \), the constant matrix \( \Gamma \in \mathbb{R}^{n \times n} \) is the inner coupling matrix, and the coupling matrix \( A = (a_{ij}) \in \mathbb{R}^{N \times N} \) denotes the coupling configuration of the entire network. If there is a connection between node \( i \) and node \( j \), then \( a_{ij} = a_{ji} = 1 \); else \( a_{ij} = a_{ji} = 0 \). Define the degree \( k_i = \sum_{j=1}^{N} a_{ij} = \sum_{j=1 \atop j \neq i}^{N} a_{ij}, i = 1, 2, \ldots, N \). Let the diagonal element of matrix \( A : a_{ii} = -k_i, i = 1, 2, \ldots, N \), which means the linear diffusive coupling, and \( c_{ii} = (1/k_i) \sum_{j=1 \atop j \neq i}^{N} c_{ij} a_{ij} \) for normalization.

The coupled network (1) is said to be (asymptotically) stable if

\[
x_1(t) = x_2(t) = \cdots = x_N(t) \rightarrow s(t) \quad \text{as} \quad t \rightarrow \infty
\]

where \( s(t) \in \mathbb{R}^n \) is a solution of an isolated node, satisfying

\[
\dot{s}(t) = f[s(t)].
\]
s can be an equilibrium point, a periodic orbit, or even a chaotic attractor, depending on the interests of study.

The uncertain diffusive coupling \( h_i \) will vanish when all the nodes are in the stationary manifold \( x_1 = x_2 = \cdots = x_N \). According to this, \( h_i \) can be written by \( h_i = Y_i(L \otimes I_n)X \), in which a suitable matrix \( Y_i \in \mathbb{R}^{n \times n(N-1)} \) depends on \( X = [x_1^T, x_2^T, \ldots, x_N^T]^T \), and the matrix \( (L \otimes I_n) \in \mathbb{R}^{n(N-1) \times nN} \). From the diffusive property, \( h_i = 0 \), when \( x_i = s_i \), it is obvious that \( L[1, 1, \ldots, 1]^T = 0 \). Let the error signal be \( e_i = x_i - s(i = 1, 2, \ldots, N) \). Define \( E = [e_1^T, e_2^T, \ldots, e_N^T]^T \), \( H = [h_1^T, h_2^T, \ldots, h_N^T]^T \), and \( Y = [Y_1^T, Y_2^T, \ldots, Y_N^T]^T \). Then

\[
H^T H = X^T (L^T \otimes I_n) Y^T Y (L \otimes I_n) X
= E^T (L^T \otimes I_n) Y^T Y (L \otimes I_n) E. \quad (4)
\]

\( I_n \) is a unit diagonal matrix with \( n \) diagonal elements 1.

The proposed brief will provide the stability analysis of network (1) and seek for the stabilization scheme with the pinning control strategy.

### III. MAIN RESULTS

Pinning control is used to stabilize the complex network to an equilibrium point in general situations [4], [10], [11]–[13]. Similar to the work in [13], here, we propose a pinning control strategy to stabilize the uncertain complex network (1) to a homogeneous orbit \( s(t) \) based on the V-stability tool.

#### A. V-Stability With Respect to \( s(t) \)

Here, introduce the assumption that is needed throughout the brief, where

\[
D_i = \{e_i : \|e_i\| < \alpha \}, \quad \alpha > 0. \quad (5)
\]

**Assumption 1:** There is a continuously differentiable Lyapunov function \( V(e_i) : D_i \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_+ \), \( \forall i \) satisfying \( V(0) = 0 \), such that there is a scalar \( \theta_i \) guaranteeing

\[
\frac{\partial V(e_i)}{\partial e_i} (f(s + e_i) - f(s) + \theta_i \Gamma e_i) < 0
\]

\( \forall e_i \in D_i/0, \quad i = 1, 2, \ldots, N. \quad (6) \)

Here, \( \theta_i \) is the **passivity degree** in the sense that the \( i \)-th node needs energy from outside to become stable with respect to the objective orbit \( s \) when \( \theta_i < 0 \), whereas the \( i \)-th node is already stable when \( \theta_i > 0 \). It is obvious to note that the passivity degree associated with each node is not unique in general. Optimally, for each node, the **passivity degree** is defined to be the largest one, but any permissible value of \( \theta_i \) can be taken as the **passivity degree** in practice [13].

Now, consider the following Lyapunov function for the whole network (1):

\[
V_N(E) = \sum_{i=1}^{N} V(e_i). \quad (7)
\]

Its time derivative along trajectory \( E \) is given by

\[
\dot{V}_N(E) = \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} (f(x_i) - f(s))
+ \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} \sum_{j=1}^{N} c_{ij} \alpha_{ij} \Gamma (x_j - s)
+ \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} h_i(x_1, x_2, \ldots, x_N)
\]

\[
< - \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} \theta_i \Gamma e_i
+ \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} \sum_{j=1}^{N} c_{ij} \alpha_{ij} \Gamma e_j
+ \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} h_i(x_1, x_2, \ldots, x_N). \quad (8)
\]

Thus, inequality (8) can be rewritten as \( \dot{V}_N(E) < M(E) \), where

\[
M(E) = \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} \theta_i \Gamma e_i
+ \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} \sum_{j=1}^{N} c_{ij} \alpha_{ij} \Gamma e_j
+ \sum_{i=1}^{N} \frac{\partial V(e_i)}{\partial e_i} h_i(x_1, x_2, \ldots, x_N). \quad (9)
\]

Here, define \( D = D_1 \times D_2 \times \cdots \times D_N \subseteq \mathbb{R}^{nN} \). Then, we can derive some results as follows.

**Lemma 1:**

1. Network (1) is locally asymptotically stable on the objective orbit \( s \) if \( M(E) \leq 0 \) for all \( E \in D \setminus \{0\} \).
2. Network (1) is locally exponentially stable on the orbit \( s \) if \( M(E) \leq -\mu_1 \|E\|^2, \mu_2 \|e\|^2 \leq V(e) \leq \mu_3 \|e\|^2 \), with three constants \( \mu_1, \mu_2, \mu_3 > 0 \) for all \( E \in D \). The region of attraction is

\[
\Omega = E : V_N(E) < r \quad (10)
\]

with \( r = \inf_{E \in \partial D} V_N(E) \). In the case of \( D = \mathbb{R}^{nN} \), the above stability becomes global.

In (9), the self-dynamics of the node is replaced by the corresponding passivity degree \( \theta_i \) to simplify the following stability analysis. This is a merit of the V-stability scheme, which paves the way to deduce the sufficient condition for the stability of uncertain complex networks.

#### B. Quadratic Function of \( V(e) \)

The case discussed here is that the Lyapunov function \( V(e_i) \) in Assumption 1 is selected to be a quadratic monomial as
\( V(e_i) = e_i^T Q e_i \), in which \( Q \) is a symmetric and positive-definite matrix.

For deriving the stabilization conditions, here, we give another assumption that is based on (4).

**Assumption 2:** There exist two known symmetric positive-semidefinite matrices \( M_1 \in \mathbb{R}^{N \times N} \) and \( M_2 \in \mathbb{R}^{n \times n} \) such that

\[
H^T H \leq E^T (M_1 \otimes M_2) E. \tag{11}
\]

**Theorem 2:** For each node, suppose that there exists a function \( V(e_i) = e_i^T Q e_i \), with a symmetric and positive matrix \( Q \), satisfying Assumption 1 with passivity degree \( \kappa_1 \). Thus, network (1) is locally stable over the domain \( \mathcal{D} \) if there exist two positive scalars \( a_1 \) and \( a_2 \) such that

\[
Q \Theta + \Theta^T Q \geq a_1 M_2 \]
\[
M_2 \geq a_2 Q^2 \tag{12a}
\]

\[
a_2 (a_1 (\Theta + G) + M_1) + I_N \leq 0 \tag{12b}
\]

\[
a_2 (a_1 (\Theta + G) + M_1) + I_N \leq 0 \tag{12c}
\]

are satisfied, where \( \Theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}^{N \times N} \) and \( G = (g_{ij}) = (c_{ij}a_{ij}) \in \mathbb{R}^{N \times N} \). Furthermore, the stability becomes global if \( \mathcal{D} = \mathbb{R}^{nN} \).

**Proof:** Since \( \partial V(e_i)/\partial e_i = 2e_i^T Q(e_i) \), (9) can be rewritten in the Kronecker product form as follows:

\[
M(E) = 2E^T ((-\Theta + G) \otimes Q \Theta) E + 2E^T (I_N \otimes Q) H. \tag{13}
\]

Notice that

\[
M(E) + E^T (M_1 \otimes M_2) E - H^T H
\]
\[= E^T ((-\Theta + G) \otimes (Q \Theta + \Theta^T Q) + M_1 \otimes M_2) E
\]
\[+ 2E^T (I_N \otimes Q) H - H^T H
\]
\[\leq E^T ((a_1 (\Theta + G) + M_1) \otimes M_2) E
\]
\[+ 2E^T (I_N \otimes Q) H - H^T H
\]
\[= [E^T, H^T] W \begin{bmatrix} E \\ H \end{bmatrix} \tag{14}
\]

where

\[
W = \begin{bmatrix} (a_1 (\Theta + G) + M_1) \otimes M_2 & I_N \otimes Q \\ I_N \otimes Q & -I_{nN} \end{bmatrix}.
\]

From inequalities (12b) and (12c), we can easily obtain the following inequality:

\[
- a_2 (a_1 (\Theta + G) + M_1) - I_N \otimes \left( \frac{1}{a_2} M_2 - Q^2 \right) \geq 0 \tag{15}
\]

which is equivalent to

\[
- a_2 (a_1 (\Theta + G) + M_1) \otimes \frac{1}{a_2} M_2 + I_N \otimes Q^2
\]
\[\geq - a_2 (a_1 (\Theta + G) + M_1) \otimes Q^2 + I_N \otimes \frac{1}{a_2} M_2
\]
\[\geq 0. \tag{16}
\]

Based on Schur complements, inequality (16) is equivalent to

\[
\begin{bmatrix}
-a_2 (a_1 (\Theta + G) + M_1) \otimes \frac{1}{a_2} M_2 & -I_N \otimes Q \\
-I_N \otimes Q & I_{nN}
\end{bmatrix} \geq 0. \tag{17}
\]

Therefore, \( W \leq 0 \), and it is easy to notice that \( M(E) \) is negative semidefinite with Assumption 2. The proof is, thus, completed with Lemma 1.

**Theorem 2** gives the sufficient and simple conditions (12) for determining the stability of network (1) with respect to a homogenous orbit. Here, it can be found that inequalities (12a) and (12b) always hold for sufficiently small \( a_1 \) and \( a_2 \) only if \( Q \Gamma + \Gamma^T Q > 0 \); however, they will influence the feasibility of the characteristic inequality (12c) under the V-stability scheme [13] through \( a_1 \) and \( a_2 \). Both of them can be regarded as the negative influences of uncertain couplings.

**Remark 3:** When all \( \theta_i < 0 \), it is easy to verify that inequality (12c) is infeasible. This is consistent with the fact that every node in a network needs outside energy to converge to a predefined orbit. The pinning control discussed in the following section provides such energies, which can enlarge \( \theta_i \) of controlled nodes from negative to positive, to make (12c) hold such that the coupled network (1) is stabilized to the predefined orbit \( s(t) \).

**C. Pinning Control**

To achieve the goal [see (2)], the pinning control strategy is applied on a small fraction \( \delta \) \( (0 < \delta < 1) \) of nodes in network (1). Without loss of generality, assume that the first \( m \) \((m = \left \lfloor \delta N \right \rfloor) \) nodes in network (1) are controlled by pinning controllers in the form of

\[
u_i = g_i(x_i, s), \quad i = 1, 2, \ldots, m. \tag{18}
\]

Then, the dynamic functions of the closed-loop system become

\[
\dot{x}_i = \begin{cases}
\dot{f} + h_i + \sum_{j=1}^{N} c_{ij} a_{ij} e_{j} + g_i & i = 1, 2, \ldots, m, \\
\dot{f} + h_i + \sum_{j=1}^{N} c_{ij} a_{ij} e_{j} & i = m + 1, \ldots, N.
\end{cases} \tag{19}
\]

Under the V-stability scheme, the control law aims to change the passivity degree of node self-dynamics from \( \theta_i \) to \( \theta_i + \kappa_i \), for \( i = 1, \ldots, m \), i.e.,

\[
\frac{\partial V(e_i)}{\partial e_i} (f(x_i) + g_i(x_i, s) - f(s) + (\theta_i + \kappa_i) \Gamma e_i) < 0
\]
\[\forall e_i \in \mathcal{D}_1, \ e_i \neq 0. \tag{20}
\]

Here, what we care about is the changed value of passivity degree \( \kappa_i \), instead of the style of the control law, which can arbitrarily be designed with the above inequality. Thus, according to Theorem 2, the following result can be directly established.

**Theorem 4:** The controlled network (19) is stabilized to orbit \( s \) if (12a), (12b), (20), and the following characteristic
inequality:

\[ a_2(a_1(-\Theta + G - K) + M_1) + I_N \leq 0 \]  

(21)

are satisfied, where \( K \in \mathbb{R}^{N \times N} \) is a diagonal matrix with the first \( m \) elements \( \kappa_i \) \((i = 1, 2, \ldots, m)\), and other \( N-m \) diagonal elements are all zero.

Therefore, the decisions of how many and which nodes are chosen to be pinned are converted to make sure that conditions (12a), (12b), (20), and (21) hold.

Here, we introduce one useful result.

Proposition 5: If (12a) and (12b) are satisfied and \( a_2(a_1(-\Theta + G) + M_1) + I_N \) has \( m \) nonnegative eigenvalues, then the rank of matrix \( K \) cannot be less than \( m \) when the closed-loop characteristic inequality (21) holds strictly.

Proof: Suppose the opposite situation where

\[ \text{rank}(K) < m. \]  

(22)

Without loss of generality, the symmetric matrix \( a_2(a_1(-\Theta + G) + M_1) + I_N \) can be rewritten as

\[ a_2(a_1(-\Theta + G) + M_1) + I_N = P \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_N)P^T \]

with \( P \in \mathbb{R}^{N \times N} \) being an orthogonal matrix, and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0 \). Note that \( a_2(a_1(-\Theta + G) + M_1) + I_N \geq PZP^T \), with \( Z = \text{diag}(0, \ldots, 0, \lambda_{m+1}, \ldots, \lambda_N) \). From (22), it follows that \( \text{rank}(PZP^T) = \text{rank}(a_1a_2K) < N \), which is a contradiction to the fact that \( PZP^T - a_1a_2K \) is negative definite. The proof is, thus, completed.

From Proposition 5, a much easier method is obtained to decide how many controllers should be placed at the nodes at least when (12a), (12b), and (20) hold for this stabilization problem. Due to the nonlinear dynamics of the entire network, the number of the controlled nodes may be larger than \( m \), and the choice of controlled nodes is more difficult in practice.

IV. SIMULATION RESULTS

An E.R. random network with 40 different nodes and with both linear and nonlinear couplings is discussed here to verify the proposed method. The system equation is described by (1), in which \( \Gamma = \text{diag}(1, 1, 1) \), and

\[ f(x_i) = \begin{bmatrix} 10(x_{i2} - x_{i1}) \\ 28x_{i1} - x_{i2} - x_{i1}x_{i3} \\ x_{i1}x_{i2} - \frac{3}{2}x_{i3} \end{bmatrix} \]  

(23)

which is the so-called Lorenz system. The linear coupling strengths \( c_{ij} \in [0, 20] \) are set randomly.

Here, assume that all the nonlinear couplings \( h_i \) are in the form as follows:

\[ h_i = \xi_i B_i, \quad i = 1, 2, \ldots, 40 \]  

(24)

where \( B_i \) belongs to a convex set

\[ C_o = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

and \( \xi_i \) is assumed to be in the following forms:

\[ \xi_j = \sqrt{|x_{j+1} - x_{j+2.1}|x_{j+3.2} - x_{j+2.2}|} \]

\[ \xi_{j+1} = \sqrt{|x_{j+3.2} - x_{j+2.2}|x_{j+3.3} - x_{j+3.1}|} \]

\[ \xi_{j+2} = \sqrt{|x_{j+3.3} - x_{j+3.1}|x_{j+1.2} - x_{j+3.2}|} \]

\[ \xi_{j+3} = \sqrt{|x_{j+3.1} - x_{j+1.1}|x_{j+1.2} - x_{j+3.2}|} \]

\[ j = 1, 5, 9, \ldots, 4k - 3, \quad \text{for} \ k = 1, 2, \ldots, 10. \]  

(25)

Here, it should be pointed out that some information involved in the uncertain coupling terms does not belong to the known information set based on the coupling matrix \( A \).

The objective stabilization state of the network is \( \dot{X} = (x_1^T, x_2^T, \ldots, x_40^T)^T = (s^T, s^T, \ldots, s^T)^T = S^T \), where \( s \) is a Lorenz system denoted as

\[ \dot{s} = f(s). \]  

(26)

\( h_i \) and \( e_i \) will vanish when the network is stabilized, and \( M_1 \) and \( M_2 \), which satisfy the restrictive condition (11) of \( h_i \), are not too hard to obtain as \( M_1 = \text{diag}(1, 1, 1, \ldots, 1) \in \mathbb{R}^{40 \times 40} \) and \( M_2 = \text{diag}(1, 1, 1) \). The Lyapunov function \( V(E) \) is in the form of \( V(E) = E^T E \), where \( E = ((x_1 - s)^T, (x_2 - s)^T, \ldots, (x_{40} - s)^T)^T \) and \( S = (s^T, s^T, \ldots, s^T)^T \in \mathbb{R}^{120} \). The passivity degree value of each node can be obtained as \( \theta_i = -14.6 \) over the entire domain \( \mathbb{R}^3 \). Thus, the corresponding feasible domain is \( D = \mathbb{R}^{120} \).

The scalars \( a_1 \) and \( a_2 \) are chosen as 1 to satisfy (12a) and (12b). Then, the main characteristic matrix of the simulated network is

\[ C = -\Theta + G + M_1 + I_N \]  

(27)

which has one positive eigenvalue. By Proposition 5, the number of controlled nodes cannot be less than 1 to stabilize the network to the orbit \( s \).

Set the control laws as

\[ u_{12} = -575.8(x_{12} - s) \]

(28a)

\[ u_{34} = -575.1(x_{34} - s) \]

(28b)

\[ u_{40} = -626.4(x_{40} - s) \]

(28c)

which can result in \( K = \text{diag}(\kappa_i) \), with \( \kappa_{12} = 575.8, \kappa_{34} = 575.1, \kappa_{40} = 626.4 \), and the rest of \( \kappa_i \) are zero. Then, all the eigenvalues of the closed-loop characteristic matrix \( -\Theta + G - K + M_1 + I_N \) become nonpositive; particularly, (21) is satisfied. The simulation results are shown in Figs. 1 and 2.
Fig. 1. Evolution of the network with an initial state that is randomly selected from [0, 150]. (a) Entire evolution from the initial state. (b) Segment of evolution of the network after the transition phase.

Fig. 2. Synchronization errors $e_{ij}, i = 1, 2, \ldots, 40, j = 1, 2, 3$, between the state of the Lorenz system and the state of the simulated network.

The choice of the three nodes that should be controlled is made in a trial-and-error way, while the control gains are computed by optimizing the LMI method. The simulated network with pinning controllers is globally stable to an orbit of the Lorenz system in Fig. 1. Another simulation study that is not shown here due to space limitation has also shown that the given network cannot be stabilized without any pinning control, which verifies Proposition 5.

V. CONCLUSION

In the proposed brief, stabilizing a complex network with uncertain couplings to a homogenous orbit by a pinning control strategy has been discussed based on the V-stability tool, which associates the dynamics of the nodes with passivity degrees. Theorem 2 presents a sufficient condition for the stability of the uncertain network in terms of LMIs. Then, under this condition, a pinning control strategy has been developed in Theorem 4 for stabilizing the network to a homogenous orbit. To verify the results obtained in the brief, an E.R. random network that is stabilized to the orbit of the Lorenz system has been simulated in Section IV. The proposed method is useful for practically solving the analogous stabilization problem of uncertain complex networks.

REFERENCES